

Castelnuovo-Mumford regularity

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The Castelnuovo-Mumford regularity is a kind of universal bound for important invariants of standard graded K -algebras, such as the maximum degree of the syzygies and the maximum non-vanishing degree of the local cohomology modules. It has been used as a measure for the complexity of computational problems in algebraic geometry and commutative algebra. These lectures present some evidences of these interactions, with a focus on concrete invariants of the syzygies of a homogeneous ideal in the polynomial ring. Examples are provided by using CoCoA and exercises are proposed for each lesson.

One of the aspects that makes the regularity very interesting is that it can be computed in different ways. In the first two lectures we present different approaches in studying the Castelnuovo-Mumford regularity. We discuss its behaviour with respect to hyperplane sections, intersections, sums and products of ideals.

Avoiding the construction of a minimal graded free resolution, in the third lecture we shall provide effective methods for computing the Castelnuovo-Mumford regularity. Our main reference is the paper of Bayer and Stillman which is a landmark in the subject. They showed that in generic coordinates, the regularity of an ideal coincides with the regularity of its initial ideal with respect to the reverse lexicographic order. Moreover, taking advantage of the combinatorial simplicity of the generic initial ideal, in characteristic zero, they proved that the regularity is equal to the highest degree of a minimal generator. Besides the fact that this procedure does not apply when the characteristic of K is positive, it has a very high computational cost. Following the papers by Trung, Seiler, Bermejo and Gimenez, our strategy will consist of reducing, by means of a change of coordinates as sparse as possible, the computation of the Castelnuovo-Mumford regularity of an ideal to the computation of the regularity of a monomial ideal with nice combinatorial properties. This leads us to introduce a class of monomial ideals whose associated primes are all of the form (x_0, \dots, x_i) for various i .

The last two lectures are devoted to present upper bounds for the Castelnuovo-Mumford regularity in terms of simpler invariants. The simplest invariants which reflect the complexity of a graded algebra are the dimension and the multiplicity. However, the Castelnuovo-Mumford regularity can not be bounded in terms of the multiplicity and the dimension. Notice that the regularity has been used by S. Kleiman in the construction of bounded families of sheaves with given Hilbert polynomial, a crucial point in the

construction of Hilbert or Picard scheme. Here we will present a result by Kleiman in the case of equidimensional reduced graded algebras. The problem is related to the finiteness of Hilbert functions for classes of graded algebras with given multiplicity. We will briefly discuss the problem in the graded and in the local case. We will end the presentation with a series of open problems concerning the Castelnuovo-Mumford regularity, some of them of much current interest.

The titles of the lectures:

- Lecture 1:** Castelnuovo-Mumford regularity via minimal free resolutions and Hilbert functions
- Lecture 2:** Castelnuovo-Mumford regularity and the local cohomology: its behavior relative to hyperplane sections, sums, products, intersections of ideals
- Lecture 3:** Castelnuovo Mumford regularity: computational aspects
- Lecture 4:** Finiteness of Hilbert functions and Castelnuovo Mumford regularity
- Lecture 5:** Bounds on the Castelnuovo-Mumford regularity and *Open Problems*